On Accuracy of Implicit Gradient Reconstruction (IGR) Scheme

Manish Kumar Singh\textsuperscript{1,2} and N. Balakrishnan\textsuperscript{3}

\textsuperscript{1} Scientist, Council of Scientific and Industrial Research-National Aerospace Laboratories, Bangalore, India
\textsuperscript{2} Research scholar, Indian Institute of Science, Bangalore, India
\textsuperscript{3} Professor, Indian Institute of Science, Bangalore, India

*Email: mksingh@nal.res.in

Abstract: The scope of CIR scheme is extended to finite volume method by introducing a parameter $\phi$ which controls dissipation of the scheme. It is demonstrated that the $\phi$ parameter can be linked to solution reconstruction and thus second order accuracy can be achieved effectively with a formulation corresponding to a first order scheme. An objective way of determining $\phi$ based on MUSCL kind of approach is established. The proposed method which includes the effects of reconstruction, in the absence of an explicit reconstruction step, is referred to as an Implicit Gradient Reconstruction (IGR) procedure. The work establishes the second order accuracy of the proposed method on unstructured grids for 2D linear convection equation. The method can be naturally extended to any Riemann solver involving wave-by-wave flux computation and in fact provides a means to dissipate different waves differently. The work presents a demonstration of the capability of the IGR based Euler solver for 1D problems.

1. Introduction

The modified CIR (MCIR) scheme is obtained by introducing an additional parameter $\phi$ in CIR scheme that allows an explicit control on the dissipation of the scheme. This methodology has been successfully used in conjunction with kinetic flux vector splitting based meshless method to solve many practical problems of aerospace industry \cite{1, 2, 3, 4}. The present effort establishes a rational way to determining the $\phi$ parameter by linking it to the solution gradient. In the process, it is demonstrated that the framework provides a means to obtain a family of $\phi$ schemes which seamlessly integrate spatially first order upwind and second order central difference type schemes. Particularly in the context of finite volume methodology, the second order of spatial accuracy in achieved by solution reconstruction procedure. The solution reconstruction procedure involves gradient computation within the cells. It is demonstrated that the $\phi$ parameter can be linked to the solution gradients resulting from a reconstruction procedure and therefore the use of $\phi$ can be considered as an implicit way to do solution reconstruction hence this numerical scheme has been named as Implicit Gradient Reconstruction (IGR) scheme. In this work, it is ensured that procedure required to compute $\phi$ is very much simpler than a classical solution reconstruction procedure. This is better demonstrated in two dimensional applications in the next section. The IGR scheme is very simple, efficient and obviates the need to store solution gradients, leaving significantly less memory footprint as compared to classical linear reconstruction procedure. This procedure can be used in conjunction with either flux vector splitting or flux difference splitting schemes. The accuracy of this methodology is demonstrated by solving linear circular convection in two dimension and comparing results with exact solutions. Further, standard test case for Euler equations are computed in one dimensional space.

2. Methodology

Consider 1-D convection equation given by

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$ (1)

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where physical flux $f$ is given by $f = f(u)$. For linear convection equation $f = cu$. Here, the property $u$ gets convected with the speed $c$.

Figure 1 represents convection of a linear profile over a time period of $\Delta t$. The time averaged flux on the cell interface $j + \frac{1}{2}$ is given by

$$\langle F_{j+\frac{1}{2}} \rangle = \frac{F^n_{j+\frac{1}{2}} + F^{n+1}_{j+\frac{1}{2}}}{2}$$

where $F^n_{j+\frac{1}{2}} = c \left( u_j + \frac{\delta u^n_j}{2} \right)$ and $F^{n+1}_{j+\frac{1}{2}} = c \left[ u_j + \left( \frac{\Delta x}{2} - c \Delta t \right) \frac{\delta u^n_j}{\Delta x} \right] = c \left[ u_j + \left( \frac{1}{2} - \lambda \right) \delta u^n_j \right]$

So

$$\langle F_{j+\frac{1}{2}} \rangle = c \left[ u_j + (1 - \lambda) \frac{\delta u^n_j}{2} \right] \quad (2)$$

In the above, Courant number $\lambda = \frac{|c| \Delta t}{\Delta x}$ and $\delta u^n_j$ is an undivided difference representing the solution gradient in cell $j$.

The MCIR flux at the interface $j + \frac{1}{2}$ is given by

$$\left( F_{j+\frac{1}{2}} \right)_{MCIR} = cu^n_j + \frac{1}{2} \left( c - \phi_{j+\frac{1}{2}} \right) \delta u^n_j \quad (3)$$

where $\delta u^n_j = u^n_{j+1} - u^n_j$. Comparing equation (2) and (3), we have

$$\left( \delta u^n_j \right)_{MCIR} = \frac{1 - \phi_{j+\frac{1}{2}}}{1 - \lambda} \delta u^n_j \quad (4)$$

The above equation is indicative of the relationship between $\phi_{j+\frac{1}{2}}$ parameter and cell gradient. It is noteworthy to observe that for $\phi = \lambda$, the MCIR gradient reduces to the gradient corresponding to the Lax Wendroff scheme, i.e.,

$$\left( \delta u^n_j \right)_{MCIR} = \delta u^n_j = \left( \delta u^n_j \right)_{LW}$$

Similarly, for $\phi = 1$, the MCIR gradient reduces to the gradient corresponding
to the CIR scheme, i.e., \( (\delta u^p)_{MCIR} = 0.0 = (\delta u^p)_{CIR} \). This observation is very interesting in the sense that a family of \( \phi \) schemes represented by CIR scheme for \( \phi = 1.0 \) on one end and a spatially centred scheme for \( \phi = 0.0 \) on the other end, with varying gradient representation within the finite volume, can be presented in the framework provided by MCIR.

We further explore the relation between the \( \phi \) parameter and the solution gradient by the way of analysing the MUSCL approach [5]. This is achieved by equating the interfacial flux as obtained by using a classical reconstruction scheme and the MCIR scheme

\[
F_{j+\frac{1}{2}} = \frac{c + |c|}{2} \left( u_j + \frac{\delta u_j}{2} \right) + \frac{c - |c|}{2} \left( u_{j+1} - \frac{\delta u_{j+1}}{2} \right) = \frac{c + \phi_{j+\frac{1}{2}} |c|}{2} u_j + \frac{c - \phi_{j+\frac{1}{2}} |c|}{2} u_{j+1}
\]

This leads to an expression for \( \phi \) associated with the interface \( j + \frac{1}{2} \)

\[
\phi_{j+\frac{1}{2}} = 1 - \frac{1}{2} \left( 1 + \frac{c}{|c|} \frac{\delta u_j}{\delta_x u_j} - \frac{1}{2} \left( 1 - \frac{c}{|c|} \right) \frac{\delta u_{j+1}}{\delta_x u_j} \right)
\]

The above equation formally relates the \( \phi \) parameter with the undivided solution gradient. It is interesting to note that the above method chooses the gradient from the upwind cell in the computation of \( \phi \). The above expression also suggests that if the gradients used in the computation of \( \phi \) are monotonicity preserving then the resulting scheme will also be monotonicity preserving.

In two dimension (Figure 2), the interpolation of variable 'u' at interface 'J', using cell vertex and cell centroid value is given by

\[
u^i_j = u_i + \left( \frac{u_i - u_{i,j}}{|r_i - r_{i,j}|} \right) |r_i - \bar{r}_i|
\]

\[
u^j_j = u_j + \left( \frac{u_j - u_{j,i}}{|r_j - r_{j,i}|} \right) |r_j - \bar{r}_j|
\]

Then formula for \( \phi \) at interface 'J' is given by

\[
\phi_j = 1 - \left( 1 + \frac{C_{\perp} \Delta u^j_j}{|C_{\perp} \Delta u^j_j|} \right) \frac{2 \Delta u_j^j}{\Delta u_j} + \left( 1 - \frac{C_{\perp} \Delta u^j_j}{|C_{\perp} \Delta u^j_j|} \right) \frac{2 \Delta u_j^j}{\Delta u_j}
\]

where \( \Delta u_j^j = u_j^j - u_i, \Delta u_i^j = u_j^j - u_j, \Delta u_j = u_j - u_i \). The wave propagation velocity normal to the interface is given by \( C_{\perp} = \tilde{C} \hat{n}_k \) with \( \tilde{C} \) and \( \hat{n}_k \) being the wave propagation velocity and cell interface normal, respectively.
It is evident from the 2D formulation that the gradient representation is rather implicit as against the higher order classical formulations, where the gradients are computed explicitly in the reconstruction step and therefore the resulting procedure is referred to as Implicit Gradient Reconstruction (IGR) method. It is emphasized that the proposed procedure is only marginally more expensive as compared to a formal first order formulation employing the cell averaged values themselves for flux computation. The IGR flux formula for scalar equation is given by

\[ F_J = \frac{(F_i + F_j)}{2} - \frac{\phi_J}{2} \left| C_{ij} \right| (u_j - u_i) \]  

where \( F_i \) and \( F_j \) are physical flux evaluated using the cell averaged values of cell \( i \) and \( j \), respectively.

Extension of the above procedure to Euler equations is rather straightforward, with its application to individual characteristic variables and the associated wave speeds. This would amount to recovering the gradients associated with the characteristic variables \( \delta W \) from the gradients of the primitive variables using the following expressions:

\[
(\delta W)_{1D} = \begin{bmatrix}
\rho a \delta u - \delta p \\
\rho a \delta \rho - \delta p \\
\rho a \delta u + \delta p \\
\rho a
\end{bmatrix} 
\]

Once the \( \phi \) values associated with the individual characteristic variables are known, any flux formula like the one by Steger and Warming [6] or by Roe [7], can be easily modified to include the \( \phi \) parameter associated with IGR. The IGR Roe flux formula (a modified Roe flux formula), in 1D is presented below

\[
(F_{IGR,Roe})^\pm_i = \frac{F_{LL} + F_{LR}}{2} - \frac{1}{2} \sum \left( \alpha_i \phi_i | \lambda_i | \hat{r}_i \right) 
\]

where, \( F_{LL}/F_{LR} \) denotes the Euler flux normal to the interface. \( \alpha_i \) and \( \lambda_i \) denotes strength and eigenvalue of \( i^{th} \) wave, respectively, computed from Roe averaged matrix. \( \hat{r}_i \) is Eigenvector corresponding to \( i^{th} \) wave.

For IGR Steger and Warming (IGR_SW) flux formula, positive and negative part of the eigenvalues are computed as \( \lambda_i^\pm = \frac{\lambda_i \pm \phi_i | \lambda_i |}{2} \) and first order Steger and Warming flux formula is used with these eigenvalues. The eigenvalues are defined as \( \lambda_1 = u - a \), \( \lambda_2 = u \), \( \lambda_3 = u + a \) then IGR_SW flux formula is given by

\[
(F_{IGR,SW})^\pm = \frac{\rho}{2y} \left( \frac{\lambda_1^2 + 2(y - 1) \lambda_2^2 + \lambda_3^2}{(u - a) \lambda_1^2 + 2(y - 1)u \lambda_2^2 + (u + a) \lambda_3^2} \right) 
\]

3. Results:

The IGR scheme is tested on linear convection problems in one and two dimensional space. Further, the capability of IGR based Euler solvers are demonstrated on Sod shock tube problem.
3.1 1-D Linear Convection Equation:

The test 1 involves finding the solution $u$ at $t=2.0$ for LCE $u_t + u_x = 0, x \in (0,4)$ with initial profile described by $u(x,t=0) = e^{-20(x-x_0)^2}$, $x_0 = 1.0$. The Figure 3a compares results obtained by IGR and first order accurate CIR scheme with exact solution. It is evident that IGR solution closely resembles exact solution and far superior than first order CIR solution. The test 2 requires finding solution $u$ at $t = 1.5$ for LCE $u_t + u_x = 0, x \in (0,4)$ with discontinuous initial data

$$u(x,t = 0) = \begin{cases} 1, & 1 \leq x \leq 2 \\ 0, & \text{Otherwise} \end{cases}$$

(12)

Figure 3b shows how IGR scheme performs as compared to CIR scheme. The IGR scheme captures the steep gradient in the solution quite well as against the CIR scheme which shows smeared solutions spread over several cells.

![Figure 3: (a) IGR scheme applied to Test 1, linear convection of smooth data (with $x_0=1.0$) at the output time 2.0 unit (b) IGR scheme for Test 2, linear convection of discontinuous data at the output time 1.5 unit.](image)

3.2 2-D Scalar Convection

The 2-D scalar linear convection problem with rotation around point $(x = 1, y = 0)$ of smooth Gaussian profile is considered over the rectangular domain $[0, 2] \times [0, 1]$. Here, we are looking for the steady state solution of the boundary-value problem given by Equation (13).

$$\begin{aligned}
\frac{\partial u}{\partial t} + y \frac{\partial u}{\partial x} + (1-x) \frac{\partial u}{\partial y} &= 0, & 0 < x < 2, 0 < y < 1; \\
u(x,y,t=0) &= 0, & 0 < x < 2, 0 < y < 1; \\
u(x,y=0,t) &= e^{-50(x-0.5)^2}, & 0 \leq x \leq 1, t \geq 0; \\
u(x,1,t) &= 0, & 0 < x < 2, t \geq 0; \\
u(0,y,t) &= 0, & 0 < y < 1, t \geq 0; \\
u(1,y,t) &= 0, & 0 < y < 1, t \geq 0;
\end{aligned}$$

(13)
The exact solution of this problem is given by \( u = e^{-50(\sqrt{(1-x)^2+y^2-0.5})^2} \). The initial level unstructured grid 1 contains 1794 triangular cells (Figure 4a). Each subsequent level of grid is generated by diving each triangle in to four. Figure 4b shows the isovalues of \( u \) as obtained from IGR scheme on grid 3. The isolines of the numerical solutions are almost perfect circles centred at (1.0, 0.0). The numerical solutions along the line \( x = 1.0 \) are comparing very well with exact solution in Figure 5. Table 1 summarises the accuracy studies performed on a set of four grids for circular convection problem using IGR scheme. Further, The L_1 and L_2-norm errors between exact and IGR solutions corresponding to four different grids are plotted in Figure 6. It is observed that accuracy of IGR scheme is close to 2. Figure 7 shows convergence histories of L_2 norm of residual and L_1 and L_2 error norms. It is observed that both error norms converged within 20 iterations while residual reaches the value close to \( 10^{-5} \).

![Figure 4](image1.png)

Figure 4: (a) An initial level unstructured grid 1 containing ~1800 cells (b) Contour (isovalues of \( u \)) obtained from IGR scheme on grid 3

![Figure 5](image2.png)

Figure 5: Comparison between exact solution and IGR computations at \( x = 1.0 \). The IGR results are obtained using grid 3.
Table 1: Accuracy of IGR scheme in solving 2-D circular convection problem on unstructured grid

<table>
<thead>
<tr>
<th>Grid Sequence</th>
<th>Error (Unstructured grid)</th>
<th>Accuracy</th>
<th>Convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L1 Norm</td>
<td>L2 Norm</td>
<td>L1 Norm</td>
</tr>
<tr>
<td>1</td>
<td>6.737E-03</td>
<td>1.376E-02</td>
<td>9.92E-11</td>
</tr>
<tr>
<td>2</td>
<td>1.580E-03</td>
<td>3.484E-03</td>
<td>2.1</td>
</tr>
<tr>
<td>3</td>
<td>3.761E-04</td>
<td>9.479E-04</td>
<td>2.1</td>
</tr>
<tr>
<td>4</td>
<td>9.610E-05</td>
<td>2.776E-04</td>
<td>2.0</td>
</tr>
</tbody>
</table>

Figure 6: Order of accuracy of IGR scheme for 2-D circular advection problem

Figure 7: Convergence and error history of IGR scheme on grid 3 for 2-D circular advection problem

Table 2: Initial conditions for shock tube problem with $x_o = 0.3$

<table>
<thead>
<tr>
<th>$\rho_L$</th>
<th>$u_L$</th>
<th>$p_L$</th>
<th>$\rho_R$</th>
<th>$u_R$</th>
<th>$p_R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>1.0</td>
<td>0.125</td>
<td>0.0</td>
<td>0.1</td>
</tr>
</tbody>
</table>
3.3 1-D Euler Equation: The Sod Shock Tube Problem

The Sod shock tube problem is used very frequently to test the accuracy of CFD schemes. The initial conditions for this test is given in table 2. Its exact solution consists of a left rarefaction, a contact and a right running shock. Figure 8 compares solution profiles of density, velocity, pressure and specific internal energy obtained from IGR_SW, SW 1st order, SW 2nd order and exact solution. The results obtained from IGR_SW are better than 2nd order accurate classical reconstruction SW scheme. The first order SW shows smeared solution and not able to produce satisfactory results especially for contact discontinuity. Similarly, Figure 9 shows comparison of various data profile as computed by IGR_ROE, 1st order ROE and 2nd order ROE with exact solution. It is observed that ROE flux based schemes are performing better than corresponding SW schemes. The IGR based ROE performs better than corresponding 2nd order classical reconstruction scheme in capturing contact and shock discontinuity.

![Figure 8: Comparison of IGR_SW computational results with exact solution for shock tube problem with x₀ = 0.3. Results are compared at t = 0.2 units. A total of 100 cells are used in computational domain.](image-url)
4. Conclusion:

The scope of MCIR scheme is further extended by relating \( \phi \) parameter with undivided solution gradients based on MUSCL approach. The resulting scheme, named as Implicit Gradient Reconstruction (IGR) scheme, is tested on linear scalar convection problems in one and two dimensions. It is observed that accuracy of IGR scheme is close to 2 for linear convection problem in 2-D on unstructured grids. The capability of IGR based Euler solvers in one dimensional space is demonstrated by testing it on Sod shock tube problem. It is observed that IGR_SW produces results comparable to classical reconstruction based SW scheme. Further, IGR_ROE performs better than corresponding 2nd order classical reconstruction scheme in capturing contact and shock discontinuity.
References


